MATHEMATICAL MODEL FOR THE EVOLUTIONARY DYNAMICS OF INNOVATION IN CITY PUBLIC TRANSPORT SYSTEMS

Hernán Darío Toro-Zapata
Associate professor
PhD Candidate in Automatic Engineering
Universidad del Quindío – Armenia, Colombia
ORCID: https://orcid.org/0000-0001-5519-9182
hdtoro@uniquindio.edu.co

Gerard Olivar-Tost
PhD. Professor.
Department of Mathematics and Statistics. Universidad Nacional de Colombia Sede Manizales - Colombia
ORCID: http://orcid.org/0000.0003-1862-4842
golivart@unal.edu.co

ABSTRACT.

In this study, a mathematical model is formulated and studied from the perspective of adaptive dynamics (evolutionary processes), in order to describe the interaction dynamics between two city public transport systems: one of which is established and one of which is innovative. Each system is to be influenced by a characteristic attribute; in this case, the number of assumed passengers per unit it that can transport. The model considers the proportion of users in each transport system, as well as the proportion of the budget destined for their expansion among new users, to be state variables. Model analysis allows for the determination of the conditions under which an innovative transportation system can expand and establish itself in a market which is initially dominated by an established transport system. Through use of the adaptive dynamics framework, the expected long-term behavior of the characteristic attribute which defines transport systems is examined. This long-term study allows for the establishment of the conditions under which certain values of the characteristic attribute configure coexistence, divergence,

or both kinds of scenarios. The latter case is reported as the occurrence of evolutionary ramifications, conditions that guarantee the viability of an innovative transport system. Consequently, this phenomenon is referred to as the origin of diversity.

Keywords.

Technological Change; Public Transportation Investment; Public Transportation; Simulation Modeling

Introduction

The city of Bogotá, Colombia is on the cusp of becoming one of the new world megacities. While in 1960, only seven megacities existed, by 2010, this number had increased to 27, and by 2020, it is projected that this number will grow to 37. In this growth process, cities cannot ignore fundamental aspects of their own economic and demographic development, or the complex network of interactions generated thereby (Kennedy, Stewart, Ibrahim, Facchini, & Mele, 2014). One fundamental question is the relationship between population growth, demographic development, and public transport infrastructure. Bogotá, in particular, is going through a key decision-making moment regarding the possibility of incorporating a metro system as one of its leading forms of transport. In contrast, the current mass-transit system, Transmilenio, operates using articulated buses. There is a latent need to respond to the question: under what conditions could a mass-transport system invade, expand in the market, and coexist with current, established city transport systems, in the long term? This type of question is closely related to others studied from the standpoint of evolutionary biology, and which have permitted the development of adaptive dynamics as a useful mathematical theoretical framework for the study of these questions (Baccini & Brunner, 2012).

The formation of new species, called speciation, is one of the central points of evolutionary theory. It occurs through the genetic and phenotypic divergence of populations of the same species, which adapt to different environmental niches, either within the same, or in different habitats. In allopatric speciation, two populations are geographically separated by natural or artificial barriers, while in parapatric speciation, the two populations evolve toward geographic isolation, through the exploitation of different environmental niches in contiguous habitats. In either of these two cases, geographical isolation constitutes an exogenous cause of speciation, instead of an evolutionary sequence (Dercole, & Rinaldi, 2008; Butlin, Galindo & Grahame, 2008). On the other hand, sympatric speciation considers a population in a single geographical location. As such, it is disruptive selection that exerts selection pressures, which favor extreme characteristics over average characteristics. This phenomenon may result, for example, from competition for alternative environmental niches, in which specializing may be more advantageous than being a generalist. Consequently, the population divides into two groups which are initially similar, but which later diverge on separate evolutionary paths (branches), each driven by their own mutations, undergoing what is called evolutionary branching (Butlin et al., 2008; Doebeli, & Dieckmann, 2000).

Human evolution shows empirical evidence of this evolutionary phenomenon. Humans form part of the hominidae family, which includes great apes (bonobos, chimpanzees, gorillas, and orangutans) and other extinct humanoid species. Since Darwin and the publication of The Descent of Man (1871), countless fossils have been found and dated, which show that humans and great apes shared a common ancestor approximately six or seven million years ago. The causes of the evolutionary branching which led to humans are a source of great debate. However, one of the most intriguing potential causes is the evolution of articulated language, thanks to fine control of the larynx or

the mouth, which is regulated by a particular gene (Dercole, & Rinaldi, 2008; Lai, Fisher, Hurst, Vargha-Khadem, & Monaco, 2001).

Generally speaking, the basic units capable of evolution through innovation and competition processes are not limited to living organisms. Multiple examples of evolutionary branching can, in fact, be found in material products, ideas, and social norms (Dercole, Dieckmann, Obersteiner, & Rinaldi, 2008; Dercole, Prieu, & Rinaldi, 2010; Landi, & Dercole, 2016). In particular, commercial products are replicated each time that a product is bought, and services each time they are used. They go extinct whenever they are abandoned by users. Thus, improved versions are occasionally introduced, which are characterized by small innovations. These interact in the market with the prior, established versions. Said interactions are usually competitive, and involve rivalry between products from both the same and different categories.

With the information discussed up to this point, it is possible to respond to the question of what constitutes the theory of adaptive dynamics. In general, it is a theoretical backdrop which originates in evolutionary biology, and links demographic dynamics to evolutionary changes. It further permits the description of evolutionary dynamics in the long term, considering innovations to be small and rare events (Dercole, & Rinaldi, 2008; Dieckmann & Law, 1996; Geritz, Metz, Kisdi, & Meszéna, 1997; Geritz, Meszéna, & Metz, 1998). This theory focuses on the evolutionary dynamic of quantitative adaptation attributes in the long term, and disregards genetic details, through the use of asexual demographic models. Among the most relevant aspects is that it recognizes interactions as the driving evolutionary force, and considers feedback between evolutionary change and the forces of selection experienced

by agents (Dercole, & Rinaldi, 2008; Dercole, & Rinaldi, 2010; Doebeli, & Dieckmann, 2000).

Model description

In this investigation, the question of whether conditions exist for the origin of diversity in a competitive market, among the principal public transport systems (TS) in a city, is addressed from the perspective of adaptive dynamics. Additionally, the average number of passengers transported per unit is considered to be a characteristic attribute of each TS. The model proposed here allows for determination of the innovative TS fitness function. *Invasion conditions* are established therefrom in a market dominated by a conventional TS. Later, based on theory, the *canonical equation of adaptive dynamics*, which reveals the long-term behavior of the characteristic attribute and its impact on the TS market, is determined and studied. Finally, a scenario, in which *evolutionary branching* occurs, is simulated. This phenomenon is called the *origin of diversity*, as it implies that the market can be diversified. On the other hand, a scenario in which *terminal points* occur during attribute evolution, in the case that diversification is not possible, is also presented.

Consider a city with an established transport system, which is characterized by a particular attribute, u_1, which is assumed to be positive and associated with the average number of passengers who are transported in each mobile unit. Denote $x_1 = x_1(t)$, with $0 \leq x_1 \leq 1$ the proportion of people who adopt the transport system characterized by attribute u_1. Suppose that a TS innovation occurs, which corresponds to some technological modification which physically affects the established TS, characterized by the value of attribute u_1, and leads to the appearance of an innovative TS characterized by the value of attribute u_2. In general, it is assumed that the innovation is small, and will have a minimal effect, which permits the interaction between transport systems to

occur below the same conditions, and on the same market platform. The innovative TS gives rise to a small proportion of users $x_2 = x_2(t)$ who compete with the established TS. Explicitly, the fourth-dimension system will exist as follows:

$$\dot{x}_1 = [\alpha(u_1)y_1 - \delta(u_1)](1 - x_1 - c(u_1, u_2)x_2)x_1$$

$$\dot{y}_1 = l(u_1)(1 - y_1) - \epsilon(u_1)\alpha(u_1)x_1y_1$$

$$\dot{x}_2 = [\alpha(u_2)y_2 - \delta(u_2)](1 - x_2 - c(u_2, u_1)x_1)x_2$$

$$\dot{y}_2 = l(u_2)(1 - y_2) - \epsilon(u_2)\alpha(u_2)x_2y_2.$$

In this case, the $y_i = y_i(t)$ state variable, for $i = 1$ or 2, represents the proportion of the budget invested for TS expansion, such that $0 \leq y_i \leq 1$. This model goes by the name *resident – innovative system.* Note that, for the model characteristics, must be satisfied that $0 \leq x_1 + x_2 \leq 1$.

Table 1. Description of study variables and of the coefficients used in the model

State description	
x_i	Proportion of people who use system i
y_i	Proportion of the budget available to the expansion of system i
Parameter description	
u_i	Value of the characteristic attribute which describes TS i
$\alpha(u_i)$	Rate of instant TS i adoption
$\delta(u_i)$	Rate at which TS i is abandoned by users
$l(u_i)$	Rate of investment in new resources for the expansion of TS i
$\epsilon(u_i)$	TS i efficiency of "converting" the investment into new users
$c(u_i, u_k)$	Rate of interaction between systems i and k.

On the other hand, $c(u_i, u_k)$ is the interaction rate between systems *i* and *k*. A number of situations are then obtained:

- If $c(u_i, u_k) > 1$, inter-system competition prevails over intra-system competition. A simple example of this is that, if system *i* corresponds to a city taxi system, while system *k* corresponds to a public bus system, then $c(u_i, u_k) > 1$ implies that taxi competition with buses is stronger than the competition between the taxis themselves.

- If $0 \leq c(u_i, u_k) \leq 1$, then intra-system competition prevails over inter-system competition. Returning to the public taxi and bus example, in this scenario, competition between the taxis themselves is stronger than competition between taxis and buses. Particularly, $c(u_i, u_k) = 0$ indicates that there is no interaction between the two transport systems, and $c(u_i, u_k) = 1$ indicates that the interaction between the two transport systems is symmetrical, or affects both equally.

- If $c(u_i, u_k) < 0$, the interaction between transport systems does not correspond to competition, but rather cooperation, a situation which can describe integrated TSs.

In order to numerically study the previous system, it is considered that the proportion in which new resources are invested for transport system expansion is $l(u_i) = l$, that the TS efficiency to "convert" the investment into new users is given by $\epsilon(u_i) = \epsilon$, and that the rate at which the TS is abandoned by users $\delta(u_i) = \delta$ are constants for $i = 1,2$. On the other hand, it has been assumed that the rate of instant adoption depends on characteristic attribute u, through the function:

$$\alpha(u) = a \exp\left(-\frac{1}{2a_1^2} \ln^2\left(\frac{u}{a_2^2}\right)\right).$$

For a TS characterized by attribute u, the $\alpha(u)$ rate makes perfect sense when x_1 is small, and has no competition from other transport systems (Dercole et al., 2008). A maximum of a occurs when $u = a_2^2$, in order to indicate the value of the attribute which is easiest to absorb. On the other hand, for a transport system with a very low or very high number of users, $\alpha(u)$ tends to cancel out with sensitivity controlled by a_1. Suppose that $a > 0$ and $a_1, a_2 \in \mathbb{R}$ (see Figure 1-left).
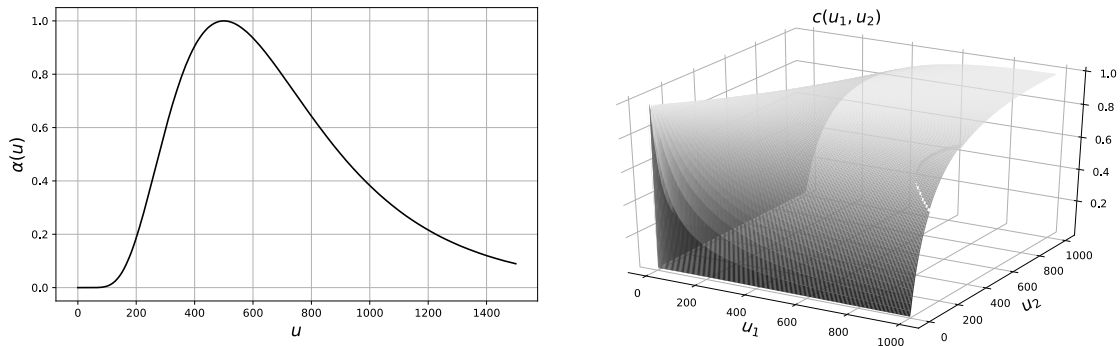

Figure 1. Left: Function $\alpha(u)$ chart for parameters $a = 1$, $a_1 = 0.5$, $a_2 = 22.36$. Right: Chart of function $c(u_1, u_2)$, for parameters $f_1 = 0.96$ y $f_2 = 2$.

Additionally, the interaction rate between TSs is represented in the following form:

$$c(u_1, u_2) = \exp\left(\frac{\ln^2 f_1}{2f_2^2}\right) \exp\left(-\frac{1}{2f_2^2} \ln^2\left(\frac{f_1 u_1}{u_2}\right)\right)$$

Observe that the interaction rate between TSs $c(u_1, u_2)$ depends on the u_1/u_2 reason, and tends toward zero when said radius tends toward zero, or when it tends toward infinity, which reflects that TSs which are very different compete weakly. A graphic representation of the function is shown in Figure 1-right (Dercole et al., 2008). If $f_1 > 1$, the TSs that move the greatest average of passengers tend to have a competitive advantage. On the other hand, if $0 <$

$f_1 < 1$, the TSs that move a lower average number of passengers will be those which have the advantage. A large f_2 value implies that very different TSs also compete strongly. When $f_1 = 1$, competition between TSs is symmetrical.

At the time in which innovation occurs, the city established TS its assumed to be in a nontrivial equilibria $\bar{x}_1(u_1) \text{ and } \bar{y}_1(u_1)$. In other words, it is assumed that this equilibria is LAS. When the resident – innovative system is studied, it may be of interest to determine the conditions under which the innovative TS of attribute u_2 can "invade" the market. For this, stability conditions at the equilibria: $E_1 = (\bar{x}_1(u_1), \bar{y}_1(u_1), 0, 1)$, must be studied. The zero and one values in the last two coordinates of E_1 indicate that the innovative TS has not yet entered the market, and that the entirety of the budget is available for investment. In order to determine local stability, a small disruption is created around it, and the behavior of the linear system associated is studied (Perko, 2013).

The values selected for simulations correspond to the belief that innovation involves a TS that is in conditions to transport a higher number of passengers. For this reason, it has been assumed that the value of the established TS attribute is $u_1 = 200$, and that the value of the innovative TS attribute is $u_2 = 800$. Although this value is far above the current capacity of the Transmilenio's bi-articulated buses, it is well below the capacity of other mass TSs. For example, a three-car train from the Medellin, Colombia metro has the capacity to transport up to 1220 passengers at a time.

A simulation of the transport systems is shown in Figure 2, both before and after innovation. The curve shown as a dashed line is the simulation of the resident system before the innovation ($x_2(t) = 0$ and $y_2(t) = 1$, for all t), respectively, for the x_1 proportion of users (left), and for the proportion of budget y_1. Once the innovation occurs, the innovative TS enters the market

(dash-dot line), which competes with the established TS (solid line). Observe that Figure 2-left corresponds to *diversification*, or what here has been called the *origin of diversity*. In effect, initially, there was just one TS established in the market. After the innovation, however, both TSs can expand and coexist in the market as two transport options for users. In particular, it should be noted that $f_1 = 0.96$ implies that the TS which mobilizes a lower number of passengers has the competitive advantage. However, innovative transport is able to expand and establish itself in the market. Similarly, in Figure 2-right, a substitution scenario is shown. The only variation that has been performed with respect to the simulation of Figure 2-left is the $f_2 = 2$ value, which indicates that, in the market, the TS with the capacity of transporting a higher number of passengers is favored.
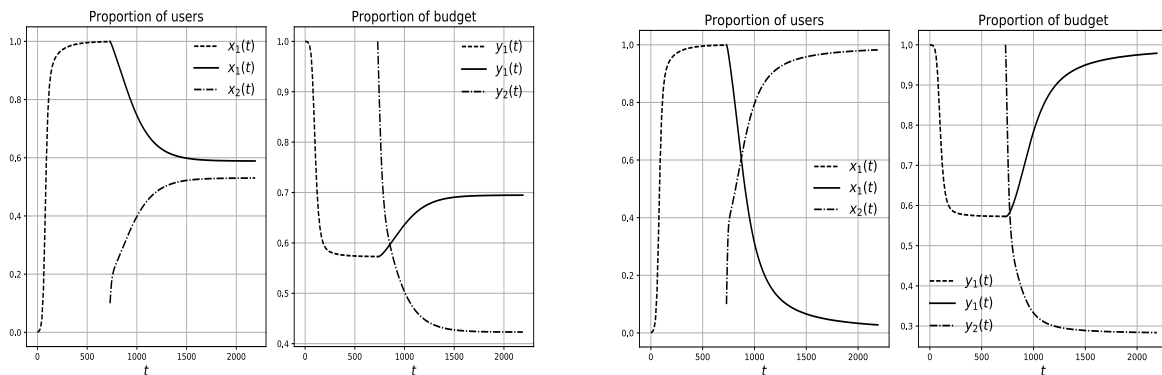

Figure 2. Diversification (left) x_1 solutions before innovation (dashed line) and after innovation (solid line) and for x_2 (dash-dot line). Right: y_1 solutions before innovation (dashed line) and after innovation (solid line) and for y_2 (dash-dot line). The following have been used: $a = 1$, $a_1 = 0.5$, $a_2 = 22.36$, $\delta = 0.1$, $\epsilon = 0.1$, $l = 0.025$, $f_1 = 0.96$, and $f_2 = 2$. The attributes are $u_1 = 200$ and $u_2 = 800$, thus, $l^*(u_1) = 0.0215$, $l^*(u_2) = 0.01184$, $R_p(u_1) = 1.8653$, and $R_p(u_2) = 6.4287$. **Substitution** (right) the same values have been used for parameters, except $f_1 = 2$.

The dynamic of the attributes, henceforth called the *evolutionary dynamic*, will help to explain the characteristics of the innovation and competition process which acts on the market. Dercole et al., 2008, succinctly describes the processes which should be considered for rigorous formulation of the *canonical equation*, which describes the evolutionary behavior (in the long term) of attribute u. This equation takes the general form:

$$\dot{u} = \frac{1}{2}\mu\sigma^2\,\bar{x}(u)\frac{\partial\lambda}{\partial u_2}\,(u,u),$$

where μ is innovation frequency, and σ^2 is the variance. In other words, the canonical equation considers the frequency with which innovations are presented in the public TS market, and the size of the variations obtained in each innovation. The $\bar{x}(u)$ value corresponds to the equilibria in which the established TS stabilizes before the innovation. On the other hand, partial derivative $\frac{\partial\lambda}{\partial u_2}\,(u,u)$, is called the *selection gradient*, and is associated with the forces of selection which are exerted from the market, by the same TS users, on the long-term dynamic of the characteristic attribute; here, $\lambda(u_1, u_2)$ is the fitness function given by one of eigenvalues of the system's Jacobian matrix at the invasion equilibria E_1 (Dercole et al., 2008).

When an evolutionary equilibria solution $\bar{u}_i$ for $i = 1$ or 2 is LAS, this means that successive innovations which replace those previous, direct attribute u toward the value of equilibria $\bar{u}_i$. It is important to consider that, in the case of diversification, or when, after innovation in the market, both TSs can coexist, each characteristic attribute will be described by a canonical equation like that described previously. The equations which correspond to this situation are not reported here, as the explicit expressions are quite long and do not significantly contribute to the discussion. However, they may be handled via symbolic calculation.

In Figure 3, the behavior of characteristic attributes u_1 and u_2 are shown, before and after innovation. It is evident that both attributes diverge in their values to different evolutionary equilibria. While the established TS from attribute u_1 is maintained below 200 passengers per mobile unit, the innovative TS progressively increases its capacity until reaching an average of over 1400 passengers transported.
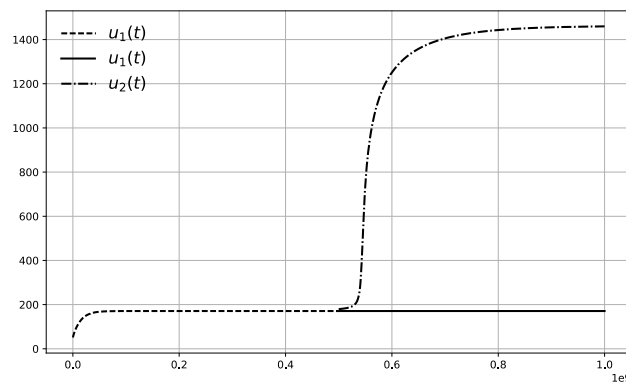

Figure 3. The solutions to the canonical equation for u_1 shown before innovation (dashed line), for u_1 canonical equations (solid line) and for u_2 (dash-dot line) after innovation. The parameters used are the same as those in Figure 9 for the diversification scenario.

Conclusions

The resident model proposed here is an initial approach to the phenomenon, the resident model permits the study of the dynamics of a city's TS in various scenarios, and learn under which conditions it may be expanded in the market, and a partial or total adoption equilibria could be found, although this would imply transporting the entire population of the city.

The innovative-resident model allows for the establishment of the conditions under which an innovative TS can invade and expand in the market. This information is obtained from study of the sign of the fitness function for

specific model coefficient expressions. Additionally, the approach through adaptive dynamics permits establishment of the long term dynamics the quantitative attribute (average number of passengers per mobile unit). The study of this evolutionary dynamic permits the classification of the evolutionary equilibrium in ramification points (diversification) or terminal points (those in which the evolution definitively halts), like the points where substitution takes place.

Particularly in the case of diversification, with the functions defined in this study, and for the values of the parameters considered, it was observed that the established TS should maintain a low number of users transported (< 200 passengers per unit), while the innovative TS should attain a high number of users transported (> 1400 passengers per unit). The above indicates that, in a scenario of coexistence between the two transport systems, it is necessary for each one of them to use a different strategy, in regards to the number of passengers that they decide to transport. One of them should focus on mobile units with few passengers, while the other system should focus on mobile units which can transport passengers massively.

Diversification is impossible when both transport systems use the same strategy. For example, if both TSs design a strategy that permits them to transport over 1400 passengers per unit, the effect would be that the innovative TS would absorb all users and substitute the established TS.

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